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Improved Dynamic Flexibility Method for Extracting Constrained Modes from Free Test Data

De-Wen Zhang*
Beijing Institute of Structure and Environment Engineering, 100076 Beijing, People's Republic of China
and
Fu-Shang Wei†
Kaman Aerospace Corporation, Bloomfield, Connecticut 06002

The dynamic flexibility method presented can be used to extract constrained structural modes from free test data. Under certain conditions the original dynamic flexibility method is not effective: 1) when the constrained structural test frequency moves into higher-order free–free analytical frequency range and 2) when the test frequency moves closer to any value $\lambda_{h,s}$ of interesting in the higher-order analytical frequencies of the free structure. The former is the test frequency range of a constrained structure larger than that of a free structure. The largest test frequency $\omega_{\max}$ of a constrained structure is higher than the smallest frequency $\lambda_{h,k+1}$ in the higher-order analytical frequencies of a free structure. Under these two separate situations a power series used in the original dynamic flexibility method is diverging, which leads to an invalid method. To solve these problems properly, an improvement of the dynamic flexibility method is proposed. The kernel technique of this improvement is a "hybrid shifting frequency" procedure. From the numerical results it is found that the improved method is better than the old method for all conditions.

Nomenclature

$A_0, A_1, \ldots$	=	matrices to be determined
$F(\omega)$	=	dynamic flexibility matrix
$F^{\Delta}(\omega^*)$	=	shifted dynamic flexibility matrix
K	=	symmetric semi-definite position stiffness matrix
K^*, K^{Δ}	=	shifted stiffness matrices
$\bar{K}$	=	transformed stiffness matrix
M	=	symmetric positive definite mass matrix
$\bar{M}$	=	transformed mass matrix
$\Delta\lambda, \Delta\lambda_1, \Delta\lambda_2$	=	shifting value of eigenvalues
Λ_h	=	higher-order free–free analytical frequencies
$\Lambda_h^*, \Lambda_h^{\Delta}$	=	shifted higher-order free–free analytical frequencies
Λ_k	=	lower-order free–free test frequencies
$\Lambda_k^*, \Lambda_k^{\Delta}$	=	shifted free–free test frequencies
$\lambda_{h,k+1}$	=	smallest value of higher-order free–free analytical frequencies
$\lambda_{h,k+1}^*, \lambda_{h,k+1}^{\Delta}$	=	shifted smallest higher-order free–free analytical frequencies
$\lambda_{h,s}$	=	sth higher-order free–free analytical frequencies
Φ_h	=	higher-order free–free analytical modes
Φ_k	=	lower-order free–free test modes
φ_c	=	extracted constrained structural modes
Ψ	=	transformed matrix
ω	=	extracted constrained structural frequency
$\| \|$	=	second norm
$\| \|$	=	absolute value

Subscripts and Superscripts

b	=	number of boundary degrees of freedom
i	=	number of internal degrees of freedom
k	=	number of free–free test modes
$*, \Delta$	=	shifting quantities

Introduction

THE technical difficulties encountered while transforming the boundary conditions during a modal test are always challenging problems for structural engineers. These kinds of problem are primarily divided into two categories. The first category is to obtain free modes from a constrained structural test data.[1–5] The second category is to extract constrained modes from free structural test results.[6–9] It is the opinion of both authors that the second type of task is easier than the first type because the measured modal data of the second type are more than that of the first type. Normally, the modal information at the boundary has been already measured. The need for the inclusion of certain static tests and measurements of other dynamic parameters implies that the first type of test is more difficult. For example, measurements of modal reaction forces at restrained degrees of freedom sometimes are required. These measurements present new challenges to both instrumentation and test engineers. As the size of a structure becomes larger, a constrained modal test is more easily realized. Thus the first type of task shall also be investigated heavily.

For the second type of task, Rubin's residual flexibility representation[10] was modified and shown in Ref. 8 to establish a residual flexibility method for extracting constrained modes from free–free test data. In addition, a practical dynamic flexibility method based on a combination of test and analysis information was also proposed in Ref. 9. An obvious merit of these methods is that the test modes, when constraining any portion of a structure, can be extracted using methods presented in Refs. 8 and 9 after conducting the test of the free structure. This is very beneficial to modify the portion of the restrained boundary that needed to be changed. However, there are two practical situations that are not described in Ref. 9: 1) ω striding into Λ_h situation and 2) $\omega \approx \lambda_{h,k+1}$ or $\omega \approx \lambda_{h,s}$ situation. Under condition 1 the test frequency range of a restrained structure is larger than that of a free structure. The highest test frequency $\omega_{\max}$ of a constrained structure is larger than that of the smallest frequency $\lambda_{h,k+1}$ in the higher-order analytical frequencies Λ_h of a free structure. In condition 2 $\lambda_{h,s}$ is a frequency value of interest in the higher-order analytical frequencies Λ_h of free structure including $\lambda_{h,k+1}$. Under these two situations a power series existing in the original method[9] becomes nonconvergent; therefore, the original dynamic flexibility

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*Professor and Senior Research Fellow, First Research Division, P.O. Box 9210. Member AIAA.

†Principal Engineer and Professor, Engineering and Development Department. Senior Member AIAA.

method cannot be utilized. The tactics for solving these problems are provided in this paper. The method is a hybrid shifting frequency technique. This technique can make the power series of the original dynamic flexibility (DF) method converge very quickly, such that the precision of the extracting constrained stuctural modes is improved significantly when these two situations occur during the analysis.

Dynamic Flexibility Method

Extracting of Constrained Structural Modes

If a constrained structure vibrates harmonically at the natural frequency ω, there is an harmonic force $R_b(\omega)$ existing at the constrained boundary. When the constrained boundary condition is released, the structure will be simulated equivalently to a free structure that is excited by a boundary force $R_b(\omega)$. The undamped equation of motion for this free structure is

$$\begin{bmatrix} M_{\text{ii}} & M_{\text{ib}} \\ M_{\text{bi}} & M_{\text{bb}} \end{bmatrix} \begin{Bmatrix} \ddot{X}_i \\ \ddot{X}_b \end{Bmatrix} + \begin{bmatrix} K_{\text{ii}} & K_{\text{ib}} \\ K_{\text{bi}} & K_{\text{bb}} \end{bmatrix} \begin{Bmatrix} X_i \\ X_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_b(\omega) \end{Bmatrix} \tag{1}$$

$R_b(\omega) = R_b e^{j\omega t}$, $X = x e^{j\omega t}$ and t represents time. Equation (1) can be rewritten as

$$\left(\begin{bmatrix} K_{\text{ii}} & K_{\text{ib}} \\ K_{\text{bi}} & K_{\text{bb}} \end{bmatrix} - \omega \begin{bmatrix} M_{\text{ii}} & M_{\text{ib}} \\ M_{\text{bi}} & M_{\text{bb}} \end{bmatrix} \right) \begin{Bmatrix} x_i \\ x_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_b \end{Bmatrix} \tag{2}$$

in which ω indicates ω^2 in order to simplify the writing. This simplified description is applied to all of the later sections of this paper. Equation (2) is reduced to

$$(K - \omega M)x = R \tag{3}$$

The eigenequation for the free structure is written as

$$(K - \lambda M)\varphi = 0 \tag{4}$$

Let Λ_k and Φ_k be the free test eigenpairs as well as Λ_h and Φ_h be the higher-order analytical eigenpairs from Eq. (4). The displacement response x shown in Eq. (3) can be represented as a linear combination of the lower-order test modes Φ_k and the higher-order analytical modes Φ_h of the free structure:

$$x = \Phi_k q_k + \Phi_h q_h = [\Phi_k, \Phi_h] \begin{Bmatrix} q_k \\ q_h \end{Bmatrix} = \Phi q \tag{5}$$

Substituting Eq. (5) into Eq. (3) gets

$$\left(\begin{bmatrix} \Lambda_k & 0 \\ 0 & \Lambda_h \end{bmatrix} - \omega \begin{bmatrix} I_k & 0 \\ 0 & I_h \end{bmatrix} \right) \begin{Bmatrix} q_k \\ q_h \end{Bmatrix} = \begin{Bmatrix} \Phi_k^T R \\ \Phi_h^T R \end{Bmatrix} \tag{6}$$

in which $I_k = \Phi_k^T M \Phi_k$ and $I_h = \Phi_h^T M \Phi_h$. From the second row of Eq. (6), q_h can be found. Then the obtained expression of q_h is returned into Eq. (5) to yield

$$x = \Phi_k q_k + F_h(\omega) R \tag{7}$$

Here $F_h(\omega) = \Phi_h (\Lambda_h - \omega I_h)^{-1} \Phi_h^T$. Equation (7) is partitioned as

$$\begin{Bmatrix} x_i \\ x_b \end{Bmatrix} = \begin{bmatrix} \Phi_{\text{ki}} \\ \Phi_{\text{kb}} \end{bmatrix} q_k + \begin{bmatrix} F_{h,\text{ii}} & F_{h,\text{ib}} \\ F_{h,\text{bi}} & F_{h,\text{bb}} \end{bmatrix} \begin{Bmatrix} 0 \\ R_b \end{Bmatrix} \tag{8}$$

From the second row of Eq. (8) and based on $x_b = 0$ condition, one knows that

$$R_b = -F_{h,\text{bb}}^{-1} \Phi_{\text{kb}} q_k \tag{9}$$

Equation (7) is rewritten as

$$x = \Phi_k q_k + [F_{h,i} \quad F_{h,b}] \begin{Bmatrix} 0 \\ R_b \end{Bmatrix} = \Phi_k q_k + F_{h,b} R_b = \Psi q_k \tag{10}$$

where

$$\Psi = \Phi_k - F_{h,b} F_{h,\text{bb}}^{-1} \Phi_{\text{kb}} \tag{11}$$

$$F_{h,i} = \begin{bmatrix} F_{h,\text{ii}} \\ F_{h,\text{bi}} \end{bmatrix}, \qquad F_{h,b} = \begin{bmatrix} F_{h,\text{ib}} \\ F_{h,\text{bb}} \end{bmatrix} \tag{12}$$

Combining Eqs. (3) and (10) yields the following eigenequation for extracting constrained structural modal parameters from a free structural test data:

$$[\bar{K}(\omega) - \omega \bar{M}(\omega)] q_k = \Psi^T R = 0 \tag{13}$$

in which ω is the restrained structural frequency to be extracted and stands for eigenvalue for short, and

$$\bar{K}(\omega) = \Psi^T K \Psi, \qquad \bar{M}(\omega) = \Psi^T M \Psi \tag{14}$$

The solution of the nonlinear eigenequation (13) is ω and q_k. From Eqs. (10) and (11) one can find the following constrained structural modes as shown in Eq. (15):

$$\phi_c = \left(\Phi_{\text{ki}} - F_{h,\text{ib}} F_{h,\text{bb}}^{-1} \Phi_{\text{kb}}\right) q_k \tag{15}$$

Computation of Dynamic Flexibility Submatrix $F_{h,b}(\omega)$

The coefficient matrix $(K - \omega M)$ of a forced vibration equation (3) for a free structure is defined as a dynamic stiffness matrix in comparison to a static stiffness matrix K in a static equilibrium equation. Correspondingly, the calculation of the entire dynamic flexibility matrix for a forced vibration system as shown in Eq. (3) is

$$F(\omega) = (K - \omega M)^{-1}$$

$$= (K^* - \omega^* M)^{-1} \tag{16a}$$

$$= \Phi(\Lambda^* - \omega^* I)^{-1} \Phi^T \tag{16b}$$

$$= \Phi_k \left(\Lambda_k^* - \omega^* I_k\right)^{-1} \Phi_k^T + \Phi_h \left(\Lambda_h^* - \omega^* I_h\right)^{-1} \Phi_h^T$$

$$= \Phi_k \left(\Lambda_k^* - \omega^* I_k\right)^{-1} \Phi_k^T + A_0 + \omega^* A_1 + \omega^{*2} A_2 + \cdots \tag{16c}$$

Equation (16b) is a spectral expression of the dynamic flexibility. From the definition one knows that

$$F_h(\omega) = A_0 + \omega^* A_1 + \omega^{*2} A_2 + \cdots \tag{17}$$

in which $K^* = K - \Delta\lambda M$, $\Lambda^* = \Lambda - \Delta\lambda I$, $\omega^* = \omega - \Delta\lambda$, $\Lambda_k^* = \Lambda_k - \Delta\lambda I_k$, and $\Lambda_h^* = \Lambda_h - \Delta\lambda I_h$. The governing equations for solving matrices $A_p (p \geq 0)$ as shown in Eq. (16c) are[9]

$$K^* A_0 = I - M \Phi_k \Phi_k^T \tag{18a}$$

$$K^* A_p = M A_{p-1}, \qquad p \geq 1 \tag{18b}$$

Because K^* is a function of $\Delta\lambda$, one can only use a unique $\Delta\lambda$; otherwise, K^* needs to be decomposed many times, which is time consuming. In addition, one does not require the calculation of all column vectors in $A_p (p \geq 0)$ and only needs to calculate those column vectors in $A_p (p \geq 0)$ corresponding to the boundary degrees of freedom (DOFs) because one needs $F_{h,b}(\omega)$ only. Thus from Eq. (17) one knows that

$$F_{h,b}(\omega) = \bar{A}_0 + \omega^* \bar{A}_1 + \omega^{*2} \bar{A}_2 + \cdots \tag{19}$$

Here submatrix $\bar{A}_p (p \geq 0)$ is formed by the column vectors in $A_p (p \geq 0)$ corresponding to the boundary DOFs.

It is necessary to point out that the convergent rate of the ω^* power series of Eq. (16c) is equivalent to that of the geometric series:

$$\frac{1}{\lambda_{h.k+1}^* - \omega^*} = \frac{1}{\lambda_{h.k+1}^*} \left[1 + \frac{\omega^*}{\lambda_{h,k+1}^*} + \left(\frac{\omega^*}{\lambda_{h,k+1}^*} \right)^2 + \cdots \right] \tag{20}$$

From Eq. (16a) one knows that the DF equation adopts the existing formal shifting frequency technique.[11] This technique can accelerate the convergence of the series shown in Eq. (20), but the accelerated effect is not obvious. Here $\lambda^*_{h,k+1} = \lambda_{h,k+1} - \Delta\lambda$, and $\lambda_{h,k+1}$ is the smallest frequency in the higher-order analytical frequencies Λ_h of the free structure.

Solution of Nonlinear Eigenequation

The initial iterative value $\omega_{(0)}$ of Eq. (13) is given by using the search procedure together with the Sturm sequence as shown in Ref. 9. The following fixed-point weighted formula[12] is employed to obtain the initial iterative vector $q_{k(0)}$:

$$[\bar{K}(\omega_{(0)}) - \omega_{(0)}\bar{M}(\omega_{(0)})]z_{(p)} = z_{(p-1)}, \qquad p = 1, 2, \ldots s \quad (21a)$$

$$q_{k(0)} = z_{(s)} \quad (21b)$$

Then $\omega_{(0)}$ and $q_{k(0)}$ are substituted into the following shifting Rayleigh inverse iterative process.[13] The required ω and q_k are found by

$$\{\bar{K}[\omega_{(j)}] - \omega_{(j)}\bar{M}[\omega_{(j)}]\}q_{k(j+1)} = f_{(j)} \quad (22a)$$

$$\bar{f}_{(j+1)} = \bar{M}[\omega_{(j)}]q_{k(j+1)} \quad (22b)$$

$$\omega_{(j+1)} = \omega_{(j)} + \frac{q^T_{k(j+1)}f_{(j)}}{q^T_{k(j+1)}\bar{f}_{(j+1)}} \quad (22c)$$

$$f_{(j+1)} = \frac{\bar{f}_{(j+1)}}{\left[q^T_{k(j+1)}\bar{f}_{(j+1)}\right]^{\frac{1}{2}}} \quad (22d)$$

where $j = 0, 1, 2$. When $j = 0$,

$$f_{(0)} = \bar{M}[\omega_{(0)}]q_{k(0)} \quad (23)$$

Improved DF Method

Both conditions 1 and 2 mentioned in the Introduction are special problems that have been emphatically discussed. They are described as follows.

Situation of ω Striding into Λ_h

This is a special situation normally encountered by the design engineers. This situation means $\omega > \lambda_{h,k+1}$, that is, $(\omega^*/\lambda^*_{h,k+1}) > 1$ when $\Delta\lambda < \lambda_{h,k+1}$. The geometric series shown in Eq. (20) is divergent, and so is the power series of $F_{h,b}(\omega)$. This problem is also discussed in Ref. 9 because both the formal shifting frequency technique and individual shifting procedure are employed in Ref. 9. The individual shifting method is the use of different shifting frequency value $\Delta\lambda$ for different ω. The use of the individual shifting frequency value can make the absolute value $|\omega^*/\lambda^*_{h,k+1}| < 1$. The original intention of using the formal shifting frequency method is to make the coefficient matrix K^* of Eqs. (18) a nonsingular matrix. It also maintains the sparse and banded characteristic of original singular matrix K. But the individual shifting method results in a K^* matrix being decomposed over and over again for different ω. This is because the K^* is a function of $\Delta\lambda$. To avoid this situation, a group shifting procedure is utilized in Ref. 9. The group shifting technique is the use of an identical shifting frequency value $\Delta\omega$ for all different ω. This method is simpler, but the group shifting method cannot guarantee every $|\omega^*/\lambda^*_{h,k+1}| < 1$ for any one group of ω values. To efficiently compare the improved method with the original method,[9] a unique $\Delta\lambda$ value for all ω is taken. Under this situation the original method needs to be improved.

A hybrid shifting frequency technique[14] is utilized here to solve the problem when ω striding into Λ_h. The dynamic flexibility expression $F^\Delta(\omega^*)$ of the forced vibration system is established with the hybrid shifting frequency technique. This DF expression will converge very fast and is defined as

$$F^\Delta(\omega^*) = (K^\Delta - \omega^* M)^{-1} \quad (24)$$

in which $K^\Delta = K + \Delta\lambda_1 M$ and $\omega^* = \omega - \Delta\lambda_2$. Using complete eigenpair (Λ, Φ) of a free structure, Eq. (24) can be rewritten as[14]

$$F^\Delta(\omega^*) = \Phi(\Lambda^\Delta - \omega^* I)^{-1}\Phi^T \quad (25a)$$

$$= \Phi_k\left(\Lambda^\Delta_k - \omega^* I_k\right)^{-1}\Phi^T_k + F^\Delta_h(\omega^*) \quad (25b)$$

$$= \Phi_k\left(\Lambda^\Delta_k - \omega^* I_k\right)^{-1}\Phi^T_k + A^\Delta_0 + \omega^* A^\Delta_1 + \omega^{*2} A^\Delta_2 + \cdots \quad (25c)$$

$$\begin{aligned} &= -\omega^{*-1}\Phi_1\Phi^T_1 - \omega^{*-2}\Phi_1\Lambda^\Delta_1\Phi^T_1 - \omega^{*-3}\Phi_1\Lambda^{\Delta 2}_1\Phi^T_1 - \cdots \\ &+ \Phi_2\Lambda^{\Delta-1}_2\Phi^T_2 + \omega^*\Phi_2\Lambda^{\Delta-2}_2\Phi^T_2 + \omega^{*2}\Phi_2\Lambda^{\Delta-3}_2\Phi^T_2 + \cdots \\ &+ A^\Delta_0 + \omega^* A^\Delta_1 + \omega^{*2} A^\Delta_2 + \cdots \end{aligned} \quad (25d)$$

in which $\Lambda^\Delta = \Lambda + \Delta\lambda_1 I$, $\Lambda^\Delta_1 = \Lambda_1 + \Delta\lambda_1 I$, and $\Lambda^\Delta_2 = \Lambda_2 + \Delta\lambda_1 I$. First use the relationship $(K^\Delta - \omega^* M)F^\Delta(\omega^*) = I$, and then embed Eq. (25d) into the relationship. Collecting all terms with the same power of ω^* and setting them equal to one another, the governing equation of matrices $A^\Delta_0, A^\Delta_1 \ldots$ as shown in Eq. (25c) can be written as

$$K^\Delta A^\Delta_0 = I - M\Phi_k\Phi^T_k \quad (26a)$$

$$K^\Delta A^\Delta_p = MA^\Delta_{p-1}, \qquad p \geq 1 \quad (26b)$$

The convergent rate of the power series of ω^* using Eq. (25c) is equivalent to the following geometric series:

$$\frac{1}{\lambda^\Delta_{h,k+1} - \omega^*} = \frac{1}{\lambda^\Delta_{h,k+1}}\left[1 + \frac{\omega^*}{\lambda^\Delta_{h,k+1}} + \left(\frac{\omega^*}{\lambda^\Delta_{h,k+1}}\right)^2 + \cdots\right] \quad (27)$$

From Ref. 14 for all ω values, only the unique value $\Delta\lambda_1$ is required to guarantee the matrix K^Δ nonsingular. Also $\Delta\lambda_1$ shall be as small as possible. In addition, under the condition of $\Delta\lambda_2 < \omega$ for certain value of ω, $\Delta\lambda_2$ shall be as large as possible. Also, $\Delta\lambda_2$ is different for various ω values to guarantee $\omega^* \approx 0$. Obviously, the geometric series shown in Eq. (27) converges very fast, thus the following power series shown in Eq. (28) will also converge very fast:

$$F^\Delta_h(\omega^*) = A^\Delta_0 + \omega^* A^\Delta_1 + \omega^{*2} A^\Delta_2 + \cdots \quad (28)$$

To compute unknown $F_h(\omega)$ based on known $F^\Delta_h(\omega^*)$ from both Eqs. (26) and (28), the relationship between $F_h(\omega)$ and $F^\Delta_h(\omega^*)$ needs to be addressed. Based on the definition, one knows that[14]

$$F_h(\omega) = \Phi_h(\Lambda_h - \omega I_h)^{-1}\Phi^T_h \quad (29)$$

$$F^\Delta_h(\omega^*) = \Phi_h\left(\Lambda^\Delta_h - \omega^* I_h\right)^{-1}\Phi^T_h \quad (30)$$

Using Eqs. (29) and (30), one has

$$\begin{aligned} F^\Delta_h(\omega^*) - F_h(\omega) &= \Phi_h\left[\left(\Lambda^\Delta_h - \omega^* I_h\right)^{-1} - (\Lambda_h - \omega I_h)^{-1}\right]\Phi^T_h \\ &= -\Delta\bar{\bar{\lambda}}\Phi_h\left(\Lambda^\Delta_h - \omega^* I_h\right)^{-1}(\Lambda_h - \omega I_h)^{-1}\Phi^T_h \\ &= -\Delta\bar{\bar{\lambda}}\Phi_h\left(\Lambda^\Delta_h - \omega^* I_h\right)^{-1}\Phi^T_h M\Phi^T_h(\Lambda_h - \omega I_h)^{-1}\Phi^T_h \\ &= -\Delta\bar{\bar{\lambda}}F^\Delta_h(\omega^*)MF_h(\omega) \end{aligned} \quad (31)$$

in which $\Delta\bar{\bar{\lambda}} = \Delta\lambda_1 + \Delta\lambda_2$ and the relationship $\Phi^T_h M\Phi_h = I_h$ has already been used in the analysis. Finally from Eq. (31) the governing equation for solving $F_h(\omega)$ is found as follows:

$$\begin{aligned} \left[I - \Delta\bar{\bar{\lambda}}F^\Delta_h(\omega^*)M\right]F_h(\omega) &= F^\Delta_h(\omega^*) \\ &= A^\Delta_0 + \omega^* A^\Delta_1 + \omega^{*2} A^\Delta_2 + \cdots \end{aligned} \quad (32)$$

From the requirement of Ref. 9, one only needs to find $F_{h,b}(\omega)$, which is formed by a small portion of the whole column in $F_h(\omega)$.

These columns correspond with the DOFs at the constrained boundary. Thus from Eq. (32) one can establish the governing equation of $F_{h,b}(\omega)$ as

$$\left[I - \Delta\bar{\bar{\lambda}} F_h^{\Delta}(\omega^*) M\right] F_{h,b}(\omega) = \bar{A}_0^{\Delta} + \omega^* \bar{A}_1^{\Delta} + \omega^{*2} \bar{A}_2^{\Delta} + \cdots \quad (33)$$

where the definition of $\bar{A}_0^{\Delta}, \bar{A}_1^{\Delta} \ldots$ is similar to that of $\bar{A}_0, \bar{A}_1 \ldots$.

Clearly, Eq. (33) is an accurate equation for solving $F_{h,b}$ and can give very accurate results. The coefficient matrix of Eq. (33) is a function of ω; hence, the present coefficient matrix needs to be decomposed over and over again for different ω. This process is time consuming. To save computer time, Eq. (33) becomes a k-step iterative formula:

$$F_{h,b(k)} = F_{h,b}^{\Delta} + X F_{h,b(k-1)}, \qquad k \geq 1 \quad (34)$$

Here

$$X = \Delta\bar{\bar{\lambda}} F_h^{\Delta} M \quad (35)$$

If one defines

$$F_{h,b} = [f_1, f_2, \ldots f_b] \quad (36a)$$

$$F_{h,b}^{\Delta} = \left[f_1^{\Delta}, f_2^{\Delta}, \ldots f_b^{\Delta}\right] \quad (36b)$$

then Eq. (34) can also be rewritten as

$$f_{i(k)} = f_i^{\Delta} + X f_{i(k-1)}, \qquad k \geq 1, \qquad i = 1, 2, \ldots b \quad (37)$$

in which b stands for the number of DOFs at the constrained boundary.

Based on the past experience as shown in Ref. 14, using the basic iterative formula (34) or (37) alone the required numerical precision for practical engineering design problems cannot be achieved. Therefore, an expanded accelerated iterative (EAI) algorithm presented in Ref. 14 and the accelerated iterative algorithm 2 proposed in Ref. 15 must be adopted.

The procedure for using the EAI algorithm is briefly presented here. For a certain column vector f, the initial parameter is

$$\Xi_0 = \left[\xi_0^{(1)}, \xi_0^{(2)}, \xi_0^{(3)}\right]^T = (0, 0, 0)^T \quad (38)$$

For $k = 1, 2, \ldots,$

$$u_k^{(1)} = f^{\Delta} + \xi_{k-1}^{(1)}\left[u_{k-1}^{(2)} - f^{\Delta}\right] + \xi_{k-1}^{(2)}\left[u_{k-1}^{(3)} - f^{\Delta}\right] + \xi_{k-1}^{(3)}\left[u_{k-1}^{(4)} - f^{\Delta}\right] \quad (39)$$

$$u_k^{(2)} = f^{\Delta} + X u_k^{(1)} \quad (40)$$

If $\|u_k^{(2)} - u_k^{(1)}\| / \|u_k^{(2)}\| < \sigma$, stop and let $f = u_k^{(2)}$; otherwise,

$$u_k^{(3)} = f^{\Delta} + X u_k^{(2)} \quad (41)$$

$$u_k^{(4)} = f^{\Delta} + X u_k^{(3)} \quad (42)$$

$$\beta_k = \left\{\left[u_k^{(1)} - u_k^{(2)} + f^{\Delta}\right], \left[u_k^{(2)} - u_k^{(3)} + f^{\Delta}\right], \left[u_k^{(3)} - u_k^{(4)} + f^{\Delta}\right]\right\} \quad (43)$$

$$\rho_k = \left[u_k^{(1)}, u_k^{(2)}, u_k^{(3)}\right] \quad (44)$$

$$\Xi_k = \left[\rho_k^T M \beta_k\right]^{-1} \rho_k^T M f^{\Delta} = \left[\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}\right]^T \quad (45)$$

Let $k := k + 1$, and then return to the calculation of Eqs. (39) and (40). Here σ is a small quantity, and $\sigma > 0$.

Situations of $\omega \approx \lambda_{h,k+1}$ and $\omega \approx \lambda_{h,s}$

At these conditions the power series of $F_{h,b}(\omega)$ is divergent, even using the formal shifting frequency and individual shifting procedures because there exists $(\omega^*/\lambda_{h,k+1}^*) \cong (\omega/\lambda_{h,k+1}) \approx 1$ in Eq. (20). A hybrid shifting frequency technique[14] should be used. Employing the hybrid shifting frequency, one can select various $\Delta\lambda_2$ values for different ω to make all ω^* approach zero. Thus one can realize $(\omega^*/\lambda_{h,k+1}^{\Delta}) \approx 0$ and $(\omega^*/\lambda_{h,s}^{\Delta}) \approx 0$ of the system with hybrid shifting frequency, so that $F_{h,b}^{\Delta}(\omega^*)$ of the system converges quickly.

Finally, for larger values of ω that are smaller than $\lambda_{h,k+1}$ but are not nearly equal to $\lambda_{h,k+1}$, using the improved method is also very efficient for improving the precision.

Numerical Simulation

The simulation example uses a suspension beam (free structure) with pure bending. It has eight nodes. Each node possesses two DOFs; therefore, the example beam has 16 DOFs. Fixing one end of the suspension beam, a constrained beam with 14 DOFs is generated. The suspension beam is tested directly, so that five lower-order modes $\Phi_k \in R^{16,5}$ are measured, in which there are two rigid-body modes. Both the original method[9] and the improved method are used to extract six modal parameters of the cantilever beam from five modal parameters of the suspension beam. The results are listed in Table 1. Only percentage errors of the extracted constrained structural eigenpairs are listed. These errors show a significant difference in accuracy between both the original and improved methods. A study of Table 1 reveals that the fifth frequency ω_5 of cantilever beam is closer to the smallest higher-order analytical frequency $\lambda_{h,k+1} = 0.83327E{+}7 = \lambda_6$ of the suspension beam. This matches the $\omega_5 \approx \lambda_{h,k+1}$ situation. In addition, the sixth frequency ω_6 of cantilever beam strides into the higher-order modal frequency range, that is, $\omega_6 > \lambda_{h,k+1}$.

In the calculation $\Delta\lambda = \Delta\lambda_1 = 10$ are taken. The initial values $\omega_{(0)}$ of six ω obtained by using the search procedure and the Sturm sequence are shown, respectively, as

$$\begin{array}{lll} 0.13003E{+}4, & 0.72359E{+}5, & 0.44825E{+}6 \\ 0.20763E{+}7, & 0.76158E{+}7, & 0.14275E{+}8 \end{array} \quad (46)$$

For various ω and also for different $\omega_{(j)}$ in the process of iteration, the values of ω^* always retain 0.001. When employing the fixed-point weighting formula of Eq. (21), the slippage phenomenon always happens to cause the numerical computation to fail (i.e., the overflow of computer). Thus the fixed-point weighting formula should not be utilized. The initial vector $q_{(0)} = (1, 1, \ldots 1)^T$ is utilized directly in the shifting Rayleigh inverse iterative method. The shifting Rayleigh inverse iteration can produce satisfactory convergent results using approximately four to six iterations. The EAI algorithm can generate convergent results within two to four iterations (here $\sigma = 10^{-5}$). The calculations with the first-order and the second-order approximation of $F_{h,b}(\omega)$ and $F_{h,b}^{\Delta}(\omega^*)$ have also been made. The results with the first-order approximation of $F_{h,b}(\omega)$ and $F_{h,b}^{\Delta}(\omega^*)$ are listed in the parentheses of Table 1. The percentage error shown in Table 1 is defined as

$$\omega\% = \frac{\omega_t - \omega}{\omega_t}\%, \qquad \phi_c\% = \left(\frac{\|\phi_t - \phi_c\|}{\|\phi_t\|}\right)^{\frac{1}{2}}\% \quad (47)$$

in which subscript t represents the measured values obtained in the real testing of a constrained structure.

Discussion

1) As shown in Table 1, under three $\Delta\lambda (= 10, 0.499E{+}6$ and $0.999E{+}7)$ conditions and applying the original DF method,[9] the errors of the fifth and the sixth modal parameters of the restrained structure corresponding to ω_5 and ω_6 are all very large. This is because one of the conditions is very close to the $\omega_5 \approx \lambda_{h,k+1}$ condition and the other condition belongs to the $\omega_6 > \lambda_{h,k+1}$ situation. To improve the precision of the results obtained under such two situations, we select $\Delta\lambda = 0.999E{+}7$ as a mean value of both ω_5 and ω_6. For $\Delta\lambda = 0.999E{+}7$ with $\omega_5^*/\lambda_6^* \approx 0.170/0.166 > 1$ and $\omega_6^*/\lambda_6^* \approx -0.536/0.166$ (i.e., $|\omega_6^*/\lambda_6^*| > 1$), the precision of the fifth and sixth modal parameters is still poor. If one uses the improved method presented in this paper, $\omega^* = 0.001$ (or other small quantities) can always be guaranteed for selecting various $\Delta\lambda_2$ values. Thus $\omega^*/\lambda_{h,k+1}^{\Delta} \ll 1$ can be realized for all ω. Therefore, employing the improved method can achieve satisfactory precision for both the $\omega > \lambda_{h,k+1}$ and $\omega \approx \lambda_{h,s}$ situations.

Table 1 Comparison of the methods with the different approximation of $F_h(\omega)$ and $F_h^{\Delta}(\omega^*)$[a]

			Extracted modal parameter and error of constrained structure											
			Original method[9]									Improved method		
			$\Delta\lambda = 10$			$\Delta\lambda = 0.499E+6$			$\Delta\lambda = 0.999E+7$			$\Delta\lambda_1 = 10$ $\omega^* = 0.001$		
Mode	Test frequency of free structure, λ	Test frequency of constrained structure, ω_t	ω	$\omega\%$	$\varphi_c\%$	ω	$\omega\%$	$\varphi_c\%$	ω	$\omega\%$	$\varphi_c\%$	ω	$\omega\%$	$\varphi_c\%$
1	0	$0.14538E+4$	$0.14538E+4$	$-0.271E-4$	$0.117E-4$	$0.14538E+4$	$-0.404E-7$	$0.697E-4$	$0.15418E+4$	-6.06	0.959	$0.14538E+4$	$0.190E-8$	$0.870E-9$
			$(0.14538E+4)$	$(0.188E-6)$	$(0.771E-7)$	$(0.14538E+4)$	$(-0.902E-5)$	$(0.933E-3)$	$(0.18512E+4)$	(-27.3)	(3.46)	$(0.14538E+4)$	$(0.190E-8)$	$(0.870E-9)$
2	0	$0.77138E+5$	$0.77515E+5$	-0.255	0.170	$0.77138E+5$	$-0.487E-8$	$0.107E-3$	$0.15418E+4$	98.0	115.0	$0.77318E+5$	$0.534E-8$	$0.328E-8$
			$(0.77318E+5)$	$(0.329E-4)$	$(0.552E-4)$	$(0.77318E+5)$	$(-0.152E-5)$	$(0.152E-2)$	$(0.18512E+4)$	(97.6)	(112.0)	$(0.77318E+5)$	$(0.534E-8)$	$(0.328E-8)$
3	$0.12570E+6$	$0.50365E+6$	$0.58933E+6$	-17.0	15.1	$0.50365E+6$	$0.289E-12$	$0.245E-9$	$0.52596E+6$	-4.43	6.17	$0.50365E+6$	$-0.194E-7$	$0.686E-5$
			$(0.50365E+6)$	$(0.320E-3)$	$(0.331E-2)$	$(0.50365E+6)$	$(0.162E-12)$	$(0.323E-6)$	$(0.55269E+6)$	(-9.74)	(11.7)	$(0.50365E+6)$	$(-0.194E-7)$	$(0.686E-5)$
4	$0.81001E+6$	$0.22529E+7$	$0.58933E+6$	73.8	98.7	$0.22529E+7$	$-0.610E-7$	$0.671E-3$	$0.22609E+7$	-0.359	2.46	$0.22529E+7$	$0.312E-6$	$0.949E-5$
			$(0.22528E+7)$	$(0.344E-3)$	$(0.865E-2)$	$(0.22529E+7)$	$(-0.345E-5)$	$(0.563E-2)$	$(0.22621E+7)$	(-0.412)	(2.35)	$(0.22529E+7)$	$(0.312E-6)$	$(0.949E-5)$
5	$0.23854E+7$	$0.82870E+7$	$0.58933E+6$	92.9	112.0	$0.22529E+7$	72.8	76.0	$0.13267E+8$	-60.1	78.7	$0.82869E+7$	$0.112E-2$	$0.797E-2$
			$(0.22528E+7)$	(72.8)	(76.0)	$(0.22529E+7)$	(72.8)	(76.0)	$(0.15890E+8)$	(-91.7)	(196.0)	$(0.82869E+7)$	$(0.112E-2)$	$(0.797E-2)$
6	$0.83323E+7$	$0.15349E+8$	$0.58933E+6$	96.2	93.7	$0.11294E+8$	26.4	125.0	$0.13319E+8$	13.2	73.5	$0.15348E+8$	$0.479E-2$	$0.375E-1$
			$(0.13203E+8)$	(14.0)	(139.0)	$(0.12963E+8)$	(15.5)	(137.0)	$(0.11380E+8)$	(25.9)	(83.3)	$(0.15348E+8)$	$(0.497E-2)$	$(0.394E-1)$

[a]Data in parentheses are the results obtained from the first-order approximation of $F_h(\omega^*)$ and $F_h^{\Delta}(\omega^*)$.

2) When using the original method,[9] if $\Delta\lambda$ is not carefully selected the precision of the results given by using the higher-order approximation of $F_h(\omega)$ might be even worse than those achieved from the use of the lower-order approximation of $F_h(\omega)$. For example, when $\Delta\lambda = 10$ the results are poor. There is no such problem when adopting the improved method. Employing the improved method merely requires the first-order approximation (even the zero-order approximation) of $F_h^{\Delta}(\omega^*)$ to obtain satisfactory precision of the results.

3) Considering the inherent property of the structure, it should be taken into consideration that $\Delta\lambda$ or $\Delta\lambda_1$ values should not be very large. They were selected for eliminating the singularity of K matrix. In a practical engineering application $\Delta\lambda$ or $\Delta\lambda_1$ should be as small as possible under the prerequisite of K^* or K^{Δ} being a nonsingular matrix. This will prevent the characteristic of mass M submerging the characteristic of stiffness K. To realize this, the improved method has the advantage in comparison with the original method.

4) Using the improved method, one can make ω^* retain unchange for all ω. In addition, $\bar{A}_p^{\Delta}(p \geq 0)$ given by Eq. (26) does not change. Thus $F_{h,b}^{\Delta}(\omega^*)$ acquired from Eq. (28) is not varied as ω changes. $F_{h,b}^{\Delta}(\omega^*)$ only needs to be computed once for all ω. However $F_{h,b}(\omega)$ still needs to be recomputed for different ω because $\Delta\lambda_2$ is included in $\Delta\bar{\lambda}$.

5) If the precision of the initial value $\omega_{(0)}$ shown in Eq. (46) increases, they are chosen as close as possible to real values ω_t via increasing the computer searching time. The iterative number of shifting Rayleigh inverse iteration can be decreased. Better results can be found, and the slippage phenomenon of the fixed-point weighting formula might be eluded.

6) In practical application of the improved method, the mass-orthogonality of measured modes of the free structure are needed.[16,17]

7) In Ref. 8 Rubin's method with the first-order approximation[10] is modified to extract the constrained structural modes from free test data. Rubin's first-order approximation is, in fact, a static-state solution based on the static equilibrium equation. His residual flexibility result has the contribution of unretained higher-order modes to the whole static flexibility of a structure. This contribution is defined here as the static residual flexibility. Rubin's second-order approximation is obtained based on a static governing equation by using the first-order approximate solution to approach the inertial and damping terms. In other words, the static residual flexibility does not include dynamic-state contribution of the unretained higher-order modes. It is not a function of ω. The eigenequation for extracting constrained modes established in Ref. 8 is not a nonlinear one. However the dynamic flexibility method presented in this paper is based on the governing equation of a forced vibration system. It is well known that the entire contribution of the higher-order modes can be divided into two parts: static-state (residual flexibility) and dynamic-state (residual inertia) contribution. Thus in the dynamic flexibility method both the static-state and dynamic-state contribution of the higher-order modes are retained, even if the zero-order term A_0^{Δ} in the residual terms (i.e., the power series of ω^*) corresponding to the higher-order modes also includes these two parts of contribution. So the precision of both the original DF[9] and the improved DF methods is obviously better than that of the residual flexibility method proposed in Ref. 8, which is the case particularly for the improved DF method developed in this paper. This fact can be ascertained through comparison of the percentage errors listed in this paper and Ref. 8. Also the eigenequation (13) is a nonlinear one because the dynamic flexibility matrix is a function of ω. Thus, all modes of constrained structure can be extracted theoretically. This cannot be matched by the residual flexibility method.[8] Certainly, as with the general eigenequation, accuracy of the higher-order eigenpairs of Eq. (13) should be poor. However, using the improved DF method only five free-free test modes (including two rigid-body modes) are utilized to obtain six acceptable modes and frequencies of constrained structure. In particular, precision of the sixth constrained structural eigenpair beyond test frequency bandwidth of free structure also is better than that of all eigenpairs extracted by using the residual flexibility method.[8] Note that the eigenpairs after the sixth eigenpair are not extracted in the calculation. If they are extracted,

the precision of some eigenpairs in them might also be better in comparison with the results of the residual flexibility method.

Conclusions

1) The dynamic flexibility method is a good method to extract constrained structural modes from free test data. Under two special situations the original dynamic flexibility method has a limitation and needed to be improved.

2) From the results shown in Table 1, one knows that adopting the improved method can satisfactorily resolve the special problem in which the precision of the original method either decreases or cannot be used when both $\omega > \lambda_{h,k+1}$ and $\omega \approx \lambda_{h,s}$ conditions happen.

3) From the numerical results it is found that the improved method is valid for two special situations and is better than the original dynamic flexibility method for all general situations. These show that new improved method is much better than old approach.

4) Both the improved and old methods can be employed for arbitrary constrained boundaries, such as fixed, pinning, etc.

References

[1]Przemieniecki, J. S., *Theory of a Matrix Structural Analysis*, McGraw–Hill, New York, 1968, pp. 357–359.

[2]Zhang, D. W., "A Galerkin Method for Transforming Constrained Structure Test Modes to Free Structure Test Modes," *Acta Astronautica Sinica*, Vol. 7, No. 4, 1988, pp. 64–69.

[3]Zhang, D. W., "Extraction of Free-Free Modes Using Constrained Structure Test Modes—Free Force of the Boundary Degrees of Freedom," *Proceedings of the 5th Modal Analysis and Experiment Conference*, Shanghai Jiaotong Univ., Zhangjiajie, PRC, 1988, pp. 762–769.

[4]Berman, A., "Free Body Structural System Identification Using Constrained Test Data," *Proceedings of the 8th International Modal Analysis Conference*, Society of Experimental Mechnics, Bethel, CT, 1990.

[5]Zhang, O., and Zerva, A., "Extraction of Free-Free Modes Using Constrained Test Data," *AIAA Journal*, Vol. 33, No. 12, 1995, pp. 2440–2442.

[6]Blair, M. A., and Vadlamudi, N., "Constrained Structural Dynamic Model Verification Using Free Vehicle Suspension Testing Methods," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC Structure, Structural Dynamics, and Materials Conference*, AIAA, Washington, DC, 1988, pp. 1187–1193.

[7]Admire, J. R., Tinker, M. L., and Ivey, E. W., "Mass-Additive Modal Test Method for Verification of Constrained Structural Models," *AIAA Journal*, Vol. 31, No. 11, 1993, pp. 2148–2153.

[8]Admire, J. R., Tinker, M. L., and Ivey, E. W., "Residual Flexibility Test Method for Verification of Constrained Structural Models," *AIAA Journal*, Vol. 32, No. 1, 1994, pp. 170–175.

[9]Liu, F., Zhang, D. W., and Zhang, L., "Dynamic Flexibility Method for Extracting Constrained Structural Modes from Free Test Data," *AIAA Journal*, Vol. 39, No. 2, 2001, pp. 279–284.

[10]Rubin, S., "Improved Component-Mode Representation for Structural Dynamic Analysis," *AIAA Journal*, Vol. 13, No. 8, 1975, pp. 995–1006.

[11]Zhang, D. W., "Development to Zhang/Wei's Dynamic Flexibility Method," *Journal of Structure and Environment Engineering*, Vol. 25, No. 3, 1998, pp. 21–27.

[12]Xu, Q. Y., and Luo, X. Y., "An Improved Algorithm for Solving Non-Linear Eigenvalue Problem of Dynamic Substructure," *Computational Structural Mechanics and Applications*, Vol. 4, No. 1, 1987, pp. 23–32.

[13]Bathe, K. J., and Wilson, E. L., *Numerical Method in Finite Element Analysis*, Prenlice–Hall, Upper Saddle River, NJ, 1976, pp. 436–439.

[14]Zhang, D. W., and Wei, F. S., "Dynamic Flexibility Method with Hybrid Shifting Frequency for Eigenvector Derivatives," *AIAA Journal*, Vol. 40, No. 10, 2002, pp. 2047–2052.

[15]Zhang, O., and Zerva, A., "Accelerated Iterative Procedure for Calculating Eigenvector Derivatives," *AIAA Journal*, Vol. 35, No. 2, 1997, pp. 340–348.

[16]Baruch, M., and Bar Itzhack, I. Y., "Optimal Weighted Orthogonalization of Measured Modes," *AIAA Journal*, Vol. 16, No. 4, 1978, pp. 346–351.

[17]Zhang, D. W., and Wei, F. S., "Some Practical Complete Modal Space and Equivalence," *AIAA Journal*, Vol. 35, No. 11, 1997, pp. 1784–1787.

A. Berman
Associate Editor